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Magnetization of the Ising Model on the Generalized Checkerboard Lattice

K. Y. Lin[1] **and F. Y. Wu**[1,2]

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We consider the Ising model on the generalized checkerboard lattice. Using a recent result by Baxter and Choy, we derive exact expressions for the magnetization of nodal spins at two values of the magnetic field, $H=0$ and $H=i\frac{1}{2}\pi kT$. Our results are given in terms of Boltzmann weights of a unit cell of the checkerboard lattice without specifying its cell structures.

KEY WORDS: Ising model; spontaneous magnetization; $H=i\frac{1}{2}\pi kT$; generalized checkerboard lattice.

1. INTRODUCTION

Consideration of the magnetization of two-dimensional Ising models spans a long history in the quest for exact results in statistical mechanics. In 1949 Onsager announced as a conference remark the now celebrated expression of the spontaneous magnetization for an isotropic square Ising lattice.[(1)] While Onsager never published details of his reasoning, the mystery was solved 3 years later when Yang produced a derivation in a masterpiece of mathematical *tour de force*.[(2)] Various generalizations of the Onsager–Yang formulation appeared soon thereafter, and the spontaneous magnetization has since been obtained for other Ising lattices, including the anisotropic square,[(3)] triangular,[(4,5)] honeycomb,[(6)] and Kagomé[(6,7)] lattices, and for the square lattice in a pure imaginary magnetic field $H=i\frac{1}{2}\pi kT$.[(8,9)] The most general result, which remained unproven until very recently, is a conjecture made by Syozi and Naya in 1960 on the exact form of the spontaneous magnetization for a general square lattice consisting of four independent coupling constants.[(7,10)]

[1] Physics Department, National Tsing Hua University, Hsinchu, Taiwan 30043.
[2] Permanent address: Department of Physics, Northeastern University, Boston, Massachusetts 02115.

Recently, there has been a surge of interest in the study of this long-standing problem. In 1986 Baxter demonstrated that the spontaneous magnetization of the general square lattice considered by Syozi and Naya is actually derivable from that of the simple square lattice, and is indeed given by their conjectured formula.[11] At the same time in an independent attempt to establish the Syozi–Naya conjecture, Lin and co-workers considered a more general 4–8 (bathroom tile) lattice and proceeded to deduce its spontaneous magnetization using a more traditional approach. While their initial derivaton[12] hit a snag related to the unsolved problem of treating block Toeplitz determinants, they later proposed in its place a conjectured expression for the 4–8 lattice spontaneous magnetization.[13,14] This conjecture has since been proven to be correct by Lin[15] (who considered a partially symmetric 4–8 lattice) and by Baxter and Choy[16] (who treated the most general 4–8 lattice). With these results in hand, it is now possible to extend the consideration to a generalized checkerboard Ising lattice, which we now describe. We derive in this paper closed-form expressions for the magnetization of this generalized checkerboard lattice at two values of magnetic field, $H=0$ and $H=i\frac{1}{2}\pi kT$.

The model is defined in Section 2. We show in Section 3 that the most general checkerboard Ising lattice can be realized as a 4–8 lattice, and using this realization, we derive in Section 4 the spontaneous magnetization of the nodal spins. We also obtain in Section 5 the magnetization at the magnetic field $H=i\frac{1}{2}\pi kT$.

2. THE MODEL

Consider the checkerboard Ising lattice shown in Fig. 1. The lattice consists of nodal Ising spins σ_i denoted by black dots, which form a square array, and interaction networks, denoted by shaded squares, placed in every other face of the square array. This is a very general Ising lattice and reduces, for example, to the general square lattice of Syozi and Naya if each shaded square is a simple square of four distinct coupling constants. More generally, each shaded square may be realized by a network of (internal) spins σ_α connected to the rest of the lattice at the four nodal spins. One such example is the Utiyama lattice,[17] for which each shaded square is a network consisting of a ladder containing n steps and $2n$ internal spins.

In the most general case such a network is characterized by the Boltzmann weight

$$B(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \sum_{\sigma_\alpha = \pm 1} \exp(-\beta \mathscr{H}_b) \tag{1}$$

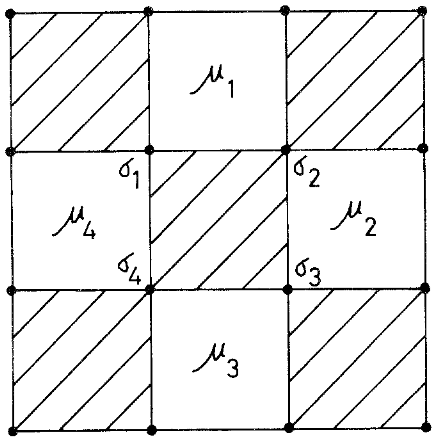

Fig. 1. The generalized checkerboard lattice. Each shaded square represents a network of internal spins connected to the rest of the lattice at four nodal spins (black dots).

where $\beta = 1/kT$, $\mathscr{H}_b$ is the Hamiltonian of the network, and σ_α refers to its internal spins. Assuming pairwise and *noncrossing* interactions, the Boltzmann weight (1) then satisfies the spin-reversal symmetry

$$B(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = B(-\sigma_1, -\sigma_2, -\sigma_3, -\sigma_4) \tag{2}$$

and the free-fermion condition[18]

$$B_1 B_2 + B_3 B_4 = B_5 B_6 + B_7 B_8 \tag{3}$$

where

$$\begin{aligned} B_1 &= B(++++), & B_2 &= B(-+-+) \\ B_3 &= B(--++), & B_4 &= B(+--+) \\ B_5 &= B(-+--), & B_6 &= B(---+) \\ B_7 &= B(+---), & B_8 &= B(--+-) \end{aligned} \tag{4}$$

The partition function of this Ising model,

$$Z = \sum_{\sigma_i = \pm 1} \prod_{\substack{\text{shaded} \\ \text{squares}}} B(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \tag{5}$$

has been evaluated by Hsue *et al.*,[19] by regarding the problem as one instance of the staggered free-fermion eight-vertex model. The one-spin correlation, or the magnetization, of a nodal spin, say σ_1 located at the upper left corner of a shaded square in Fig. 1, is taken to be

$$\langle\sigma_1\rangle = \lim_{n\to\infty} \langle\sigma_1\sigma_n\rangle^{1/2} \tag{6}$$

where σ_n is another nodal spin situated at the same corner of a shaded square but located at a distance of n squares away. Here, for $H=0$,

$$\langle\sigma_1\sigma_n\rangle = \frac{1}{Z} \sum_{\sigma_i=\pm 1} \sigma_1\sigma_n \prod_{\substack{\text{shaded}\\ \text{squares}}} B(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \tag{7}$$

The advantage of using the expression (6) as our definition of the magnetization is that it is also valid for Ising models with $H \neq 0$, provided, of course, that we include in both the numerator and the denominator of (7) appropriate additional magnetic field energies. In particular, for the magnetic field $H = i\frac{1}{2}\pi kT$, we use the identity

$$e^{i\pi\sigma/2} = i\sigma \tag{8}$$

The magnetization is then again given by (6) and (7) provided that we make the following replacements in (7):

$$\begin{aligned} &B(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \to \sigma_1\sigma_2 B(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \\ &\qquad\qquad \text{if internal spins carry no magnetic moments} \\ &B(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \to \sigma_1\sigma_2 \sum_{\sigma_\alpha=\pm 1} \left(\prod \sigma_\alpha\right) \exp(-\beta\mathscr{H}_b) \\ &\qquad\qquad \text{if all (nodal and internal) spins} \\ &\qquad\qquad \text{carry magnetic moments} \end{aligned} \tag{9}$$

Clearly, in the second case of (9), the spin-reversal symmetry (2) is satisfied only when each shaded square contains an even number of internal spins.[3] Our goal is to evaluate $\langle\sigma_1\rangle$ for both $H=0$ and $H = i\frac{1}{2}\pi kT$, given only the Boltzmann weights (4).

[3] It should also be mentioned that, in writing down (9), we have implicitly assumed that the total number of magnetic spins is an integral multiple of 4.

3. REALIZATION AS A 4–8 LATTICE

The magnetization $\langle \sigma_1 \rangle$ is invariant if we multiply the eight Boltzmann weights (4) by a common factor throughout. Furthermore, the eight weights B_α are related by the free-fermion condition (3). It follows that only six of the eight weights are independent and our goal of evaluating $\langle \sigma_1 \rangle$ is achieved if the shaded squares are realized by networks consisting of six (or more) interactions for which $\langle \sigma_1 \rangle$ is known.

One such realization for which $\langle \sigma_1 \rangle$ is known is the 4–8 lattice considered by Baxter and Choy,[16] a situation shown in Fig. 2. In this case the network represented by a shaded square consists of six distinct interactions $K_{ij} = K_1, K_2, K_3, K_4, K_{1'}$, and $K_{2'}$ and includes two internal spins σ_5 and σ_6. The Boltzmann weight (1) for a unit cell is now given by

$$B(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \sum_{\sigma_5, \sigma_6 = \pm 1} \prod B(\sigma_i, \sigma_j) \tag{10}$$

where

$$B(\sigma_i, \sigma_j) = \exp(K_{ij} \sigma_i \sigma_j)$$

It proves convenient at this point to introduce "dual" variables[20] $W_1, W_2, ..., W_8$ which are linear combinations of Boltzmann weights (10),[4]

$$W_\alpha = \sum_{\beta=1}^{8} X_{\alpha\beta} B_\beta, \qquad \alpha = 1, 2, ..., 8 \tag{11}$$

[4] The definition of W_α here is the same as that of ref. 20. However, ref. 20 contains a typographic error: the definitions of W_3 and W_4 [(9) of ref. 20] should be interchanged. (This error does not affect the results contained in ref. 20, since discussions therein were restricted to $W_3 = W_4$.)

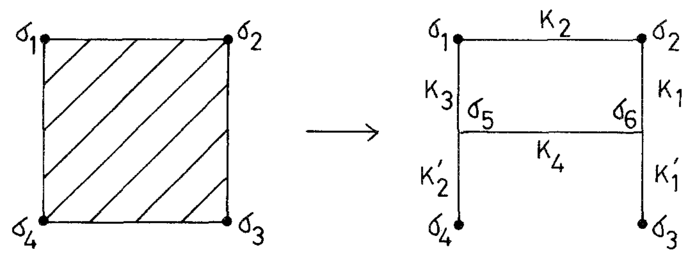

Fig. 2. Realization of the generalized checkerboard lattice as a 4–8 lattice. Each shaded square is replaced by a network consisting of six interactions and two internal spins.

Here, $X_{\alpha\beta}$ are elements of the matrix

$$X=\frac{1}{2}\begin{bmatrix} + & + & + & + & + & + & + & + \\ + & + & + & + & - & - & - & - \\ + & + & - & - & - & - & + & + \\ + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + \\ + & - & - & + & - & + & + & - \\ + & - & + & - & - & + & - & + \\ + & - & + & - & + & - & + & - \end{bmatrix} \tag{12}$$

Straightforward calculation using (4), (10), and (11) now yields[5]

$$\begin{aligned} W_1 &= \rho(1+t_1t_2t_3t_4) \\ W_2 &= \rho t_{1'}t_{2'}(t_1t_3+t_2t_4) \\ W_3 &= \rho t_{2'}(t_1t_4+t_2t_3) \\ W_4 &= \rho t_{1'}(t_1t_2+t_3t_4) \\ W_5 &= \rho t_{2'}(t_3+t_1t_2t_4) \\ W_6 &= \rho t_{1'}(t_1+t_2t_3t_4) \\ W_7 &= \rho(t_2+t_1t_3t_4) \\ W_8 &= \rho t_{1'}t_{2'}(t_4+t_1t_2t_3) \end{aligned} \tag{13}$$

where

$$\rho = 16 \cosh K_{1'} \cosh K_{2'} \prod_{i=1}^{4} \cosh K_i$$

$$t_i = \tanh K_i, \qquad t_{i'} = \tanh K_{i'}$$

4. SPONTANEOUS MAGNETIZATION

Baxter and Choy[16] have computed the spontaneous magnetization of the 4–8 lattice shown in Fig. 2. Rewriting their results, we find

$$\langle \sigma_1 \rangle_{H=0} = (1-\Omega^2)^{1/8}\, N/D \tag{14}$$

[5] To arrive at (13), it is most convenient to combine (7) of ref. 20 with (10) of this paper and adopt the standard high-temperature tanh expansion to evaluate spin sums.

where

$$1-\Omega^2=(s-\omega_1)(s-\omega_2)(s-\omega_3)(s-\omega_4)/\omega_5\omega_6\omega_7\omega_8$$
$$s=(\omega_1+\omega_2+\omega_3+\omega_4)/2$$
$$N=[(t_{1'})^{-1/2}(\omega_5\omega_7/\omega_6\omega_8)^{1/4}+(t_{1'})^{1/2}(\omega_6\omega_8/\omega_5\omega_7)^{1/4}]/2$$
$$D=2^{-1/2}\{1+(\omega_1\omega_3+\omega_2\omega_4)/[2(\omega_5\omega_6\omega_7\omega_8)^{1/2}]\}^{1/2}$$

and, aside from an overall factor which does not effect $\langle\sigma_1\rangle_{H=0}$,

$$\begin{aligned} \omega_i &= W_i, & i&=1,2,3,4\\ \omega_5 &= (t_{1'}t_{2'})^{1/2}W_7, & \omega_6&=(t_{1'}t_{2'})^{-1/2}W_8\\ \omega_7&=(t_{1'}/t_{2'})^{1/2}W_5, & \omega_8&=(t_{2'}/t_{1'})^{1/2}W_6 \end{aligned} \tag{15}$$

with W_α given by (13). The substitution of (15) into (14) now yields

$$\langle\sigma_1\rangle_{H=0}=(1-\Omega^2)^{1/8}F_1 \tag{16}$$

where

$$\begin{aligned} 1-\Omega^2 &= (-W_1+W_2+W_3+W_4)(W_1-W_2+W_3+W_4)\\ &\quad\times(W_1+W_2-W_3+W_4)(W_1+W_2+W_3-W_4)/16W_5W_6W_7W_8\\ F_1 &= [(W_5W_7)^{1/2}+(W_6W_8)^{1/2}]/\\ &\quad [W_1W_3+W_2W_4+2(W_5W_6W_7W_8)^{1/2}]^{1/2} \end{aligned}$$

Finally, the desired expression of $\langle\sigma_1\rangle_{H=0}$ in terms of the Boltzmann weights B_α is obtained by substituting (11) into (16).

In a similar fashion we can evaluate $\langle\sigma_2\rangle_{H=0}$, where σ_2 is the nodal spin located at the upper right corner of the shaded square (cf. Fig. 1). However, it is simpler to utilize a symmetry consideration. From the left–right symmetry it is clear that $\langle\sigma_2\rangle_{H=0}$ is given by (16) with the interchanges $\sigma_1\leftrightarrow\sigma_2$, $\sigma_3\leftrightarrow\sigma_4$. Observation of (4) indicates that this corresponds to the interchanges $B_5\leftrightarrow B_7$, $B_6\leftrightarrow B_8$, and hence, by (11), the interchanges of $W_3\leftrightarrow W_4$ and $W_5\leftrightarrow W_6$. Thus,

$$\langle\sigma_2\rangle_{H=0}=(1-\Omega^2)^{1/8}F_2 \tag{17}$$

where

$$F_2=[(W_6W_7)^{1/2}+(W_5W_8)^{1/2}]/[W_1W_4+W_2W_3+2(W_5W_6W_7W_8)^{1/2}]^{1/2}$$

5. MAGNETIZATION AT $H=i\frac{1}{2}\pi kT$

Consider first the case that internal spins in each shaded square of Fig. 1 carry no magnetic moment, namely the magnetic field $H=i\frac{1}{2}\pi kT$ is applied to nodal spins only. In this case we use (6) and obtain

$$\langle\sigma_1\rangle' \equiv \langle\sigma_1\rangle_{H=i\pi kT/2} = \lim_{n\to\infty} (\langle\sigma_1\sigma_n\rangle')^{1/2} \tag{18}$$

where the prime indicates that the averages are taken at $H=i\frac{1}{2}\pi kT$. Also, using the first line of (9), $\langle\sigma_1\sigma_n\rangle'$ is given by (7) with the replacement of $B(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ by

$$B'(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \sigma_1\sigma_2 B(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \tag{19}$$

Perusal of (4) shows that this corresponds to the negation of B_2, B_4, B_5, and B_7, and thus the new dual variables

$$W'_\alpha = \sum_\beta Y_{\alpha\beta} B_\beta \tag{20}$$

where

$$Y = \frac{1}{2}\begin{bmatrix} + & - & + & - & - & + & - & + \\ + & - & + & - & + & - & + & - \\ + & - & - & + & + & - & - & + \\ + & - & - & + & - & + & + & - \\ + & + & - & - & - & - & + & + \\ + & + & - & - & + & + & - & - \\ + & + & + & + & + & + & + & + \\ + & + & + & + & - & - & - & - \end{bmatrix} \tag{21}$$

is obtained by negating the second, fourth, fifth, and seventh columns of the matrix X given in (12). Comparison of (21) with (12) yields

$$\begin{aligned} &W'_1 = W_7, \quad W'_2 = W_8, \quad W'_3 = W_5, \quad W'_4 = W_6 \\ &W'_5 = W_3, \quad W'_6 = W_4, \quad W'_7 = W_1, \quad W'_8 = W_2 \end{aligned} \tag{22}$$

Arguments of Section 4 now lead to an expression for $\langle\sigma_1\rangle'$ which is the same as the rhs of (16) but with W'_i in place of W_i. Substituting (22) into this expression, we finally obtain

$$\langle\sigma_1\rangle' = (1-\Omega'^2)^{1/8} F'_1 \tag{23}$$

with

$$\begin{aligned}1-\Omega'^2 &= (-W_5+W_6+W_7+W_8)(W_5-W_6+W_7+W_8)\\&\quad\times(W_5+W_6-W_7+W_8)(W_5+W_6+W_7-W_8)/16W_1W_2W_3W_4\\F_1' &= [(W_1W_3)^{1/2}+(W_2W_4)^{1/2}]/\\&\quad[W_5W_7+W_6W_8+2(W_1W_2W_3W_4)^{1/2}]^{1/2}\end{aligned}$$

In a similar fashion we obtain $\langle\sigma_2\rangle'$ with the interchanges of $W_3 \leftrightarrow W_4$ and $W_5 \leftrightarrow W_6$ in (23). In one special case that the shaded squares contain no internal spins, we find $F_1'=1$ and (23) reproduces a recent result obtained by Lin.[21]

When all internal spins carry equal magnetic moments so that the magnetic field $H=i\frac{1}{2}\pi kT$ is applied to all spins, in our analysis we make the replacement indicated by the second line of (9). Provided that the number of internal spins in each shaded square is even, we can carry out the same analysis as before to evaluate $\langle\sigma_1\rangle'$. The resulting expression of $\langle\sigma_1\rangle'$ is then given by the rhs of (16) with W_α replaced by $W_\alpha'=\sum_\beta X_{\alpha\beta}B_\beta'$, where B_β' is

$$B'(\sigma_1,\sigma_2,\sigma_3,\sigma_4)=\sigma_1\sigma_2\sum_{\sigma_\alpha=\pm1}\left(\prod\sigma_\alpha\right)\exp(-\beta\mathscr{H}_b) \tag{24}$$

as given by (4).

In practice, however, it is often quite simple to obtain $\langle\sigma_1\rangle'$ from $\langle\sigma_1\rangle_{H=0}$ for a given spin model when the lattice is explicitly known. As remarked elsewhere,[22] one needs only to split the factor $\sigma_1\sigma_2\prod\sigma_\alpha$ appearing in (24) into a product of paired spins which are connected by interactions K_{ij}. Then $\langle\sigma_1\rangle'$ is obtained by simply converting the corresponding $\tanh K_{ij}$ into $\coth K_{ij}$ in the known expression of $\langle\sigma_1\rangle_{H=0}$. Explicit examples of evaluations of $\langle\sigma_1\rangle'$ are given in ref. 22.

6. SUMMARY

We have obtained closed-form expressions for the magnetization of nodal spins of a generalized checkerboard Ising lattice at two values of the magnetic field, $H=0$ and $H=i\frac{1}{2}\pi kT$. The resulting expressions are given by (16) and (23), respectively. Note that we have not addressed the question of computing the magnetization of internal spins within a checkerboard unit cell, since the result of this computation is obviously structurally dependent. We point out that, however, it is often possible, for a given structure of the checkerboard unit cell, to evaluate magnetizations of the internal spins by relating them to those of the nodal spins. One such example is the Union Jack lattice considered by Choy and Baxter.[23]

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